By EVEN Group

DIFFERENTIAL CALCULUS

**Function**

Function: If x and y be two variables, so related that corresponding to every value within a define domain, we get a define value of y then y is said to be the function of x defined in its domain.

Mathematically,

y= f(x), where (x, y) ∈ **R**

In this case, the variable x to which we may arbitrarily assigned in the given value which are called domains, is referred to as independent variable or argument and y is called dependent variable. All values of y are called as ranges.

Example:

(I) f(x)=|x|

y= f(x).

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| x | 0 | 1 | -1 | 2 | -2 |
| y | 0 | 1 | 1 | 2 | 2 |

**Figure 1: y=|x|**

(II) f(x)=|x|+1

y=f(x)

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| x | 0 | 1 | -1 | 2 | -1 |
| y | 1 | 2 | 2 | 3 | 3 |

**Figure 2: y=|x|+1**

(III) f(x)=|x-1|+|x+1|

y=f(x)

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| x | 0 | 1 | -1 | 2 | -1 |
| y | 2 | 2 | 2 | x | 4 |

**Figure 3: y=|x-1|+|x+1|**

**CLASSIFICATION OF FUNCTIONS:**

(I)Even Function: If f(x) is a real valued function then f(x) is an even function if the equations hold for all values of x such that x and -x are the domain of the function,

f(x)=f(-x)

or, f(x)-f(-x) =0.

Example:

(I) f(x)=x**2**, (II) f(x) = cosx, (III) f(x)=x**2**+1.

(II)Odd Function: If f(x) is a real valued function then f(x) is an odd function if the equations hold for all values of x such that x and -x are the domain of the function,

f(-x) = -f(x)

or, f(-x) + f(x) =0.

Example:

(I) f(x)=x**3**, (II) f(x) = sinx, (III) f(x)=2x+sinx.

(III)Implicit Function: Let (x, y) be two variables where the relation between x and y is expressed by an equation, say **φ** (x, y) = 0, then it is called as an implicit function.

Example:

(I) f (x, y) =x**2** + y**2**, (II) f (x, y) = x**3** + xy+ y**3.**

(IV) Explicit Function: If a function can be expressed in form as, y=f(x) and x ∈D where D ⊆ R be domain of the function then the function is called as an explicit function.

Example:

(I) y =x**3** +x+10, (II) y =

(V) Periodic Function**:** If a function f(x) is defined in a domain D then it is called as periodic function of µ when µ be the lest positive real number such, f(x+µ) =f(x) for all x ∈D. [x+ µ ∈D]

Example:

f(x)=sinx, x ∈d periodic function of 2π since 2π is a least positive number such that f(x+2π) =sin(x+2π) = sinx =f(x).

(VI) Algebraic Function**:** If a function only involves algebraic equations then it is called algebraic function.

Example:

(I)f(x)=x, (II)f(x)=x2 +x+1, (III)f(x) = .

(VII) Exponential Function: An exponential function is a function of the form where base is a real number not equal to 1 and the argument x occurs as an exponent.

Example:

(I)f(x)=, (II)f(x)= .

**PROBLEMS LIST**:

Find the domain and ranges of the following functions:

1. f(x)= 2. f(x)=

3. f(x)= 4. f(x)=

5. f(x)=

**SOLUTION:**

1**.** Given that, f(x)**= .**

As the denominator must be ≠0

therefore, x≠2

Here f(x) is defined for all values of x except x=2.

Domain of f(x) = **R –** {2}.

let, y=

⇒ xy-2y=x**2** - 4

⇒ x**2**- xy+2y- 4=0

⇒ x**2** – xy +(2y - 4) =0

Since x is real, the determinant D= b2- 4ac will be greater or equal to 0.

∴ (-y)**2** -4(-2y-4).1≥0

**⇒** y**2** - 8y +16≥0

**⇒** (y – 4)**2** ≥0

Since, (y– 4)**2** is not defined at y=4.

Range of f(x)=**R –** {4}.

(Ans)

2**.** Given that,

f(x)

As the denominator must be ≠0

therefore, x≠2 and x≠3.

Here f(x) is defined for all values of x except x=2 and x=1

Domain of f(x) = **R** – {1, 2}.

Let, y

⇒ yx**2** -3yx+2y=x-2

⇒ yx2 –(3y+1) x+2y+2=0

∴

Since x is real, the determinant D= b2- 4ac will be greater or equal to 0.

∴ (-3y-1)**2**-4y(2y+2) ≥0

⇒ 9y**2** +6y+1-8y**2**-8y≥0

⇒ y**2** -2y+1 ≥0

⇒ (y-1)**2** ≥0

Since (y-1)**2** ≥0 is not defined at y=1

let, ⇒ x= g(y)

And denominator of function x=g(y) must be  0.

therefore,2y0 ⇒ y≠0.

Range of f(x) **= R** – {1,0}

(Ans)

3. Given that,

f(x)

As the denominator must be ≠0

therefore, x≠2 and x≠ -3.

Here f(x) is defined for all values of x except x=2 and x= -3

Domain of f(x) = **R** – {2, -3}.

Let, y

⇒ yx**2** +yx-6y=x2 – 3x +2

⇒ (y-1) x2 +(y+3) x-(6y+2) =0

∴

Since x is real, the determinant D= b2- 4ac will be greater or equal to 0.

∴ (y+3)**2** -4(y-1) (-6y-1) ≥0

⇒ y**2** +6y+9 + 24y**2** -16y-8 ≥0

⇒ 25y**2** -10y+1 ≥0

⇒ (5y-1)**2** ≥0

Since (5y-1)**2** ≥0 is not defined at y=

let, x= g(y)

And denominator of function x=g(y) must be 0.

therefore, 2(y-1)0 ⇒ y≠1.

Range of f(x) **= R** – {1,}.

(Ans)

4. Given that,

f(x)

As the denominator must be ≠0

therefore, x≠2 and x≠ 3.

Here f(x) is defined for all values of x except x=2 and x= 3

Domain of f(x)**=** x ∈ (−∞, −3) ∪ (−3, −2) ∪ (−2, +∞)

Let,

⇒ yx**2** -5yx+6y=x2 +1

⇒ (y-1) x2 - 5yx +(6y-1) =0

Since x is real, the determinant D= b2- 4ac will be greater or equal to 0.

∴ (-5y)**2**-4(y-1) (6y-1) ≥0

⇒ 25y**2** -24y**2** +28y -4≥0

⇒ y**2** +28y -4≥0

⇒ (y+14)**2** – (10√2)**2** ≥0

⇒ (y+14 + 10√2) (y+14 –10√2)≥0

this inequality will be true if

(y+14 + 10√2) ≥0 and (y+14 - 10√2) ≥0

⇒ y ≥ (-14 - 10√2) and y ≥ (-14 + 10√2)

⇒ y ≥ (-14 + 10√2).

or if, (y+14 + 10√2) ≤0 and (y+14 - 10√2) ≤ 0

⇒ y ≤ (-14 -10√2) and y ≤ ( -14 + 10√2)

⇒ y ≤ (-14 -10√2).

let, x = g(y)

As denominator of function x=g(y) must be 0.

therefore, 2(y- 1)0 ⇒ y≠1.

Range of f(x)**=** y ∈ (−∞, -14 -10√2] ∪ [-14 +10√2, 1) ∪ (1, +∞)

(Ans.)

5**.** Given that,

f(x)

As the denominator must be ≠0

therefore, x≠1

Here f(x) is defined for all values of x except x=1

Domain of f(x)**=** y ∈ (−∞, 1) ∪ (1, +∞)

Let, y**=**

⇒ y (x**2** -2x+1) = 2x-1

⇒ yx2 –2yx+y-2x+1=0

⇒ yx2 –(2y+2) x+(y+1) = 0

∴ x**=**

Since x is real, the determinant D= b2- 4ac will be greater or equal to 0.

∴ (-2y-2)**2**-4y(y+1) ≥0

⇒ 4y**2** +8y+4 - 4y**2** -4y≥0

⇒ 4y+4 ≥0

⇒ 4y≥ -4

∴ y ≥ -1

let, x= g(y)

As denominator of function x=g(y) must be 0.

therefore, 2y0 ⇒ y≠0.

Range of f(x)**=** y ∈ [-1, +∞).

**LIMIT AND CONTINUITY**

Limit: A function f(x) is to tend to a limit as x tends to a if the difference between f(x) and l is less then any given positive number, however small by making x approach to given constant a.

Mathematically,

which means that | f(x) - l | is less than any given number.

Right Hand Limit: A function is said to be tend to a limit l if x

approaches the value a form right side.

Mathematically,

Sometimes  is represented by the symbol f(a+ 0) or,

f(a+ h).

Left Hand Limit: A function is said to be tend to a limit l if x

approaches the value a form left side.

Mathematically,

Sometimes is represented by the symbol f (a- 0) or,

f(a- h).

**PROBLEMS LIST**:

1. Prove by (ε- δ) the definition of limit.

2. Prove by (ε- δ) the definition of limit and find δ if **ε** =1.

**SOLUTION LIST**:

1. Let, an arbitrary positive number ε>0, however very small.

by (ε- δ) the definition of limit, For all values of x

we get,

⇒| 2a| < ε

⇒| 2a| < ε

⇒| a|< ε …….. (I)

We can determine another positive number δ depending on ε such that

⇒| a|< δ …… (II) [for all values of x]

from (I) and (II) ,

ε=δ

Where ε=δ, the value of the function f(x) = will differ from 2a by a number ε.

Hence,

(Proved)

2. Let, an arbitrary positive number ε>0, however very small.

by (ε- δ) the definition of limit, For all values of x

we get,

ε

⇒| 8|< ε

⇒| 8|< ε

⇒ 2| 2|< ε

⇒ | 2|< …….. (I)

We can determine another positive number δ depending on ε such that

⇒| 2|< δ …… (II) [for all values of x]

from (I) and (II) ,

δ=

Where δ=, the value of the function f(x) = will differ from 8 by a number ε.

Hence,

(Proved)

Again, if ε =1, δ=

(Ans)

Continuity: A function f(x) is said to be continuous for x=a, provided exists, finite and is equal to f(a).

Mathematically, f(x) is continuous at x=a, if == f(a).

**PROBLEMS LIST**:

1. A function ø(x) is defined as follows:   
ø(x)= x2 when x<1  
 =2.5 when x=1  
 =x2 + 2 when x>1  
Is ø(x) continues at x=1?

2.A function f(x) is defined as follows:   
f(x)= -x when x≤0  
 =x when 0<x<1  
 =2-x when x≥1

show that it is continuous at x=0 and x=1.

3.A function f(x) is defined as follows:

f(x)= 3+2x for ≤ x < 0  
 =3 - 2x for 0 ≤ x <  
 = -3-2x for x ≥

show that it is continuous at x=0 and discontinuous x= **.**

4.f(x)= 5x -4 for < x ≤ 1  
 =4x**2** -3x for 1< x <2  
 =3x+4 for x ≥ 2

Discuss the continuity of f(x) for x=1 and 2, and the existence of for these values.

5. f(x)= x for 0<x<1

=2-x for 1≤x≤2

= for x>2

Is f (x) continuous at x=1 and x=2 ? Does exist for these values?

SOLUTIONS:  
1. Given, ø(x) = x**2** when x<1  
 =2.5 when x=1  
 =x2 + 2 when x>1  
  
Let consider the point x=1,  
L. H. L =

=

=  
 = (1-0)**2**  
 = 1

R.H.L =

=

=+2}  
 =(1+0)2+2  
 =3  
 f(1)=2.5  
Since,L.H.L≠R.H.L≠f(1).  
Hence the function ø(x) is not continuous at x=1.

2. Given, f(x)= -x when x≤0  
 =x when 0<x<1  
 =2-x when x≥1

Let consider the point x=0,

L.H.L =

=

=   
 = 0

R.H.L =

=

=   
 = 0

f (0) = -(0) =0  
Since, L.H.L = R.H.L = f (0).  
Hence the function f(x) is continuous at x=0.

Again,

Let consider the point x=1,

L.H.L =

=

=  
 = 1-0

=1

R.H.L =

=

=   
 = 2-1+0

=1

f (1) = 2-1

=1  
Since, L.H.L = R.H.L = f (1).  
Hence the function f(x) is continuous at x=1.

(Showed)

3.Given,f(x)= 3+2x for ≤ x < 0  
 =3 - 2x for 0 ≤ x <  
 = -3-2x for x ≥

Let consider the point x=0,

L.H.L =

=

=  
 = 3+2(0-0)

=3

R.H.L =

=

=   
 = 3-2(0+0)

=3

f (0) = 3-2(0)

=3

Since, L.H.L = R.H.L = f (0).  
Hence the function f(x) is continuous at x=0.

Again,

Let consider the point x=

L.H.L =

=

=

=

= 3-3+0

=0

R.H.L =

=

=

=

= -3-3-0

= -6

f () = {-3-2}

=-3-3 = -6

Since, L.H.L ≠ R.H.L = f () .  
Hence the function f(x) is discontinuous at x=.

(Showed)

4. Given, f(x)= 5x -4 for < x ≤ 1  
 =4x**2** -3x for 1< x <2  
 =3x+4 for x ≥ 2

Let consider the value x=1,

L.H.L =

=

=  
 = 5(1-0) -4

=1

R.H.L =

=

=   
 =

=1

f (1) = 5(1) -4

= 1

Since,L.H.L=R.H.L=f(1).  
Hence the function f(x) is continuous at x=1.

Again,

Let consider the value x=2,

L.H.L =

=

=

=

= 16-6

= 6

R.H.L =

=

=   
 =

= 6+4

= 10

f (1) = 3(2) +4

= 10

Since,L.H.L=R.H.L=f(2).  
Hence the function f(x) is continuous at x=2.

Now,

Let consider the value x=1,

=

=

=

=

=

= (5+4h)

= 5

=

=

=

=

= (5)

= 5

Since = .

Hence the function exists at x = 1.

Again,

Let consider the point x=2,

=

=

=

= (3)

= 3

=

=

=

=

= (13-4h)

= 13

Since ≠ .

Hence the function f(x) does exist not at x = 2.

(Shwoed)

5. Given, f(x)= x for 0<x<1

=2-x for 1≤x≤2

= for x>2

Let, consider the value x=1,

L.H.L =

=

= = 1-0 = 1

R.H.L =

=

=   
 = 2-1-0

= 1

f (1) = 2-1

=1

Since, L.H.L = R.H.L = f (1).  
 Hence the function f(x) is continuous at x=1.

Again,

Let, consider the value x=2,

L.H.L =

=

= = 2 – 2 + 0 = 0

R.H.L =

=

=   
 =

= 2-2

= 0

f (1) = 2-2 =0

Since, L.H.L = R.H.L = f (2).  
Hence the function f(x) is continuous at x=2

Now,

Let consider the value x=1,

=

=

=

=

= -1

=

=

=

=

=

Since ≠ .

Hence the function does not exist at x = 1.

Again, Let consider the value x=2,

(-1- )

= (-1-0)

= -1

= -1

∴ = .

Therefore, the function exists at x = 2.

**Differential Calculus:**

Find the differential coefficient of:

3. (
5. ( +
6. +
7. w. r. t.
8. w. r. t.
9. w. r. t.
10. If , show that

Solution:

1. sec ()

=

=

)

=

= (Ans)

= .

= .

= .

= .

= (Ans.)

=

= )

= )

=

= = (Ans)

=

=

=

=

= + )

= +

= [ + ]

= (Ans)

1. x) consider,

= (sec tan) x = tan

= ( )

= ]

=

=

=

= + tan ()

= +

x)

= + ]

= (Ans)

1. consider,

= x + tan

=

=

=

= ]

= (Ans)

1. consider,

= x = tan

=

=

=

= 2

= 2

= (Ans)

1. consider,

= x = cos

=

=

=

=

= (Ans)

1. consider,

= x = sin

= =

=

=

= (Ans)



= .

=

= (Ans)



= [

= [] (Ans)



= [

= []

= [1 + ] (Ans)

= [

= [ +

= (Ans)



=

=

=+[] (Ans)



= [ + [

=[]+ []

=. + (Ans)

1. w.r.t

= 2 2

Considering,

y = =

z = = 2

=

=

= 1 (Ans)

1. w.r.t

x = tan

=

=

=

=

=

=

y =

=

z =

=

=

= (Ans)

1. w.r.t

Consider,

y =

z =

=

=

=

= ()[] (Ans)

1. f(x) =

f’(x) = []

= [2ln]

f’(x) = [2ln ]

f’ (0) = . [2ln]

=

(proved)

**Taylor’s Theorem:** If *(a + h)* be a function of the variable *h* such that it can be expanded in ascending powers of *h* and this expansion be differentiable with respect to *h* in any number of times then,

**. . . . . .**

**Proof:**

Consider a function –

*… … …* eqn. (1)

Differentiate with respect to *h*

*… … …* eqn. (2)

*… … …* eqn. (3)

*… … …* eqn. (4)

Put *h* = 0 in all eqn.

⇒

⇒

Now put the value of in eqn. (1)

*… … …*

Let, *a + h = x ⇒ h = x – a*

*… … …*

(proved)

**Expand in Taylor’s series:**

1. ,

Solution:

*…*

*… … …*

*… … …*

(Ans.)

2. ,

Solution:

*… … …*

*… … …*

*… … …*

*… … …*

(Ans.)

**Maclaurin’s Theorem:** If *f(x)* be a function of the variable x such that it can be expanded in ascending power of x and this expansion be differentiable with respect to x in any number of times then,

**. . . . . . . . . . . . .**

**Proof:**

Consider a function–

*… … …* eqn. (1)

Differentiate with respect to *x*

*… … …* eqn. (2)

*… … …* eqn. (3)

*… … …* eqn. (4)

Put *x* = 0 in all eqn.

⇒

⇒

The same can be written,

Now put the value of in eqn. (1)

*. . . . . .* *. . . . . . .*

(proved)

**Leibnitz’s Theorem:** If *u* and *v* are two functions of x, each possessing derivatives up to order, then the derivative of their product,

***(uv)n = unv + nc1un-1v1 + nc2un-2v2 + … … + ncrun-rvr + … … + uvn***

Where the suffixes of u and v denote the order of differentiations of *u* and *v* with respect to x.

**Proof:**

Let y = uv

By actual differentiation, we have

*y1 = u1v + uv1*

*y2 = u2v + 2u1v1 + uv2 = u2v + 2c1u1v1 + uv2*

*y3 = u3v + 3u2v1 + 3u1v2 + uv3*

*= u3v + 3c1u2v1 + 3c2u1v2 + uv3*

The theorem is thus seen to be true when *n* = 2 and 3.

Let us assume, therefore, that

*(uv)n = unv + nc1un-1v1 + nc2un-2v2 + … … + ncrun-rvr + … … + uvn*

∴ differentiating,

*yn+1 = un+1v + (nc1 + 1) unv1 + (nc2 + nc1) un-1v2 + …. + (ncr + ncr-1) un-r+1vr + uvn+1*

Since, *ncr + ncr-1 = n+1cr and nc1 + 1 = n+1c1*

∴ *yn+1 = un+1v + n+1c1 unv1 + n+1c2un-1v2 + …. + n+1crun-r+1vr + …. uvn+1*

Thus, if the theorem holds for *n* differentiations, it also holds for *n+1*. But it is probed to hold for 2 and 3 differentiations; hence it holds for four, and so on, and thus the theorem is true for every positive integral value of *n*.

**Successive Differentiation:**

1. If prove that,

Solution:

Given,

……(1)

Differentiating eqn (1) with respect to x,

Or, . . . . . . . . . . . (2)

Differentiating eqn (2) n times with respect to x,

Or,

[Proved]

2. If show that,

Solution:

Given, …….(1)

Differentiating equation (1) with respect to x, (2 times)

Or,

Or,

Or,

Or,

Or, . . . . . . . . . . . (2)

Differentiating equation (2) n times with respect to x with the help of Leibnitz theorem,

Or,

Or,

[Showed]

3. , prove that,

Solution: Given,

. . . . . . . . . .. . . (1)

Differentiating equation (1) with respect to x, (2 times)

Or,

Or, [From (1)]

Or,

. . . . . . . . . . .(2)

Differentiating equation (2) with respect to x n times by Leibnitz theorem, we get,

Or,

Or,

[Proved]

4. If then show that

Solution: Given

. . . . . . . . . . . (1)

Differentiating equation (1) with respect to x, (2 times)

Or,

Or,

Or,

Or,

Or,

Or, []

Or, . . . . . . . . . (2)

Differentiating equation (2) n times with respect to x by the help of Leibnitz's theorem,

Or,

Or,

Or,

[Showed]

5. If then show that,

Solution: Given,

. . . . . . . . . . (1)

Differentiating equation (1) with respect to x, (2 times)

, using (1)

Or,

Or,

Or,

Or,

Or,

Or, []

Or, . . . . . . . . . . (2)

Differentiating equation (2) n times with respect to x by the help of Leibnitz theorem,

Or,

Or,

Or,

[Showed]

6. show that

Solution:

Given, ........(i)

Differentiating equation (i) with respect to x (4 times),

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or, [ From equation (i)]

(showed).

7. show,

Solution:

Given, ......(i)

Differentiating equation (i) with respect to x, (2 times),

Or,

Or,

Or,

Or,

Or,

Or,

Or,

(showed)

8. If , show that

Solution:

Given, …..........(i)

Differentiating equation (i) with respect to x (4 times),

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or,

Or, [ From equation (i)]

[showed]

**Rolle’s Theorem:** Let a function f(x) be a real valued function in interval [a, b] such that,

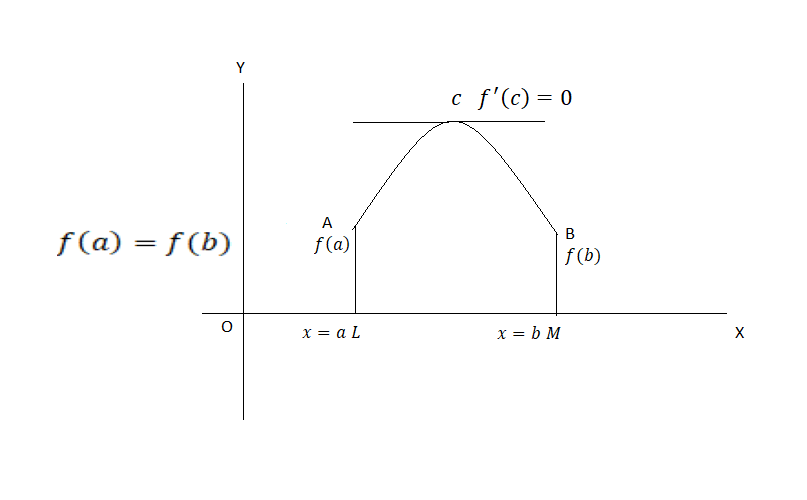
(i) f(x) is continuous in closed interval [a, b]

(ii) f(x) is differentiable in open interval (a, b)

(iii) f(a) = f(b)

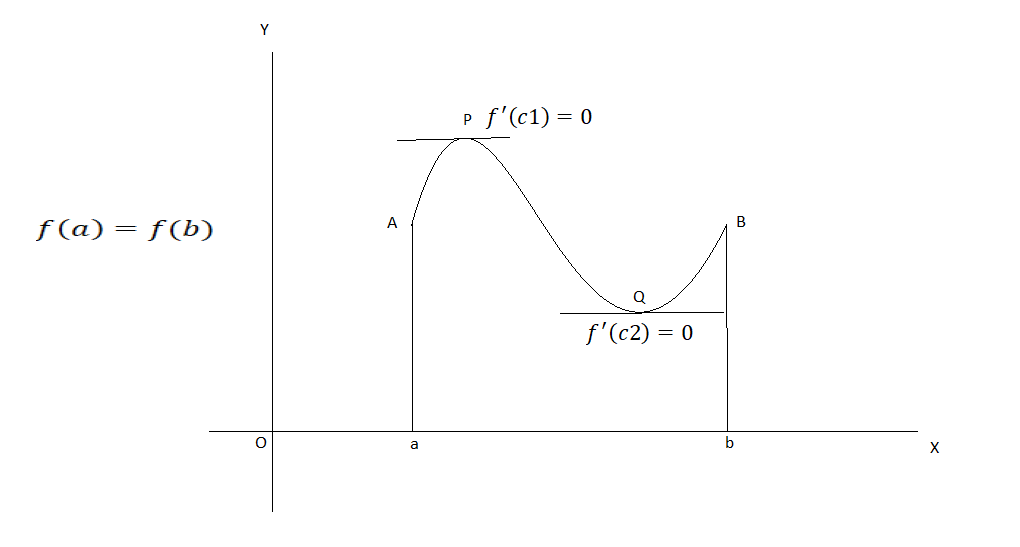
Then there exist at least one-point c ϵ (a, b) such that .

Geometrical Interpretation:



Let l, M be the points on the number axis representing the real numbers a, b respectively. We draw the graph of the function y = f(x) and let A, B be the points in it corresponding to L, M respectively, that is, LA = f(a) and MB = f(b).

From the condition (i) of Rolle’s theorem, we say that the graph is a continuous curve between the points A and B; the condition (ii) says that the curve has tangents at every point between A and B and the third condition implies that LA = MB.



Now, f(c) is the gradient of the tangent of the curve at x = c. By Rolle’s theorem vanishes at least once between x = a and x = b. Geometrically we say that we get at least one-point C on the graph between A and B such that the tangent at C is parallel to .

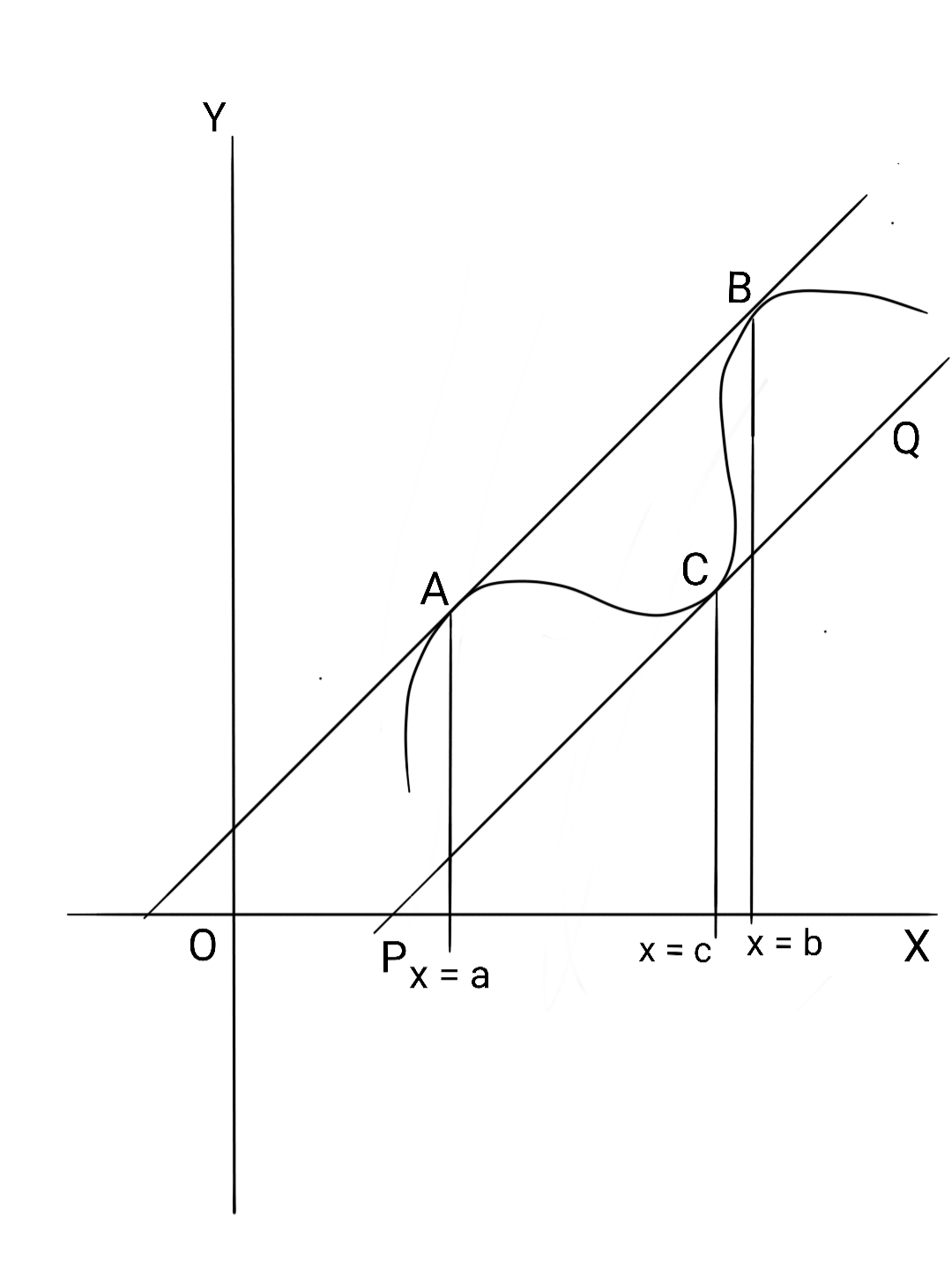
**Lagrange’s Mean Value Theorem:** Let, f(x) be defined in [a, b] such that,

(i) f(x) is continuous in [a, b]

(ii) f(x) is differentiable in (a, b)

Then, there exist at least one-point c ϵ (a, b) such that,

Geometrical Interpretation:



Let A and B are two point on the graph of f(x) corresponding to x = a and x = b respectively. Then coordinates of A and B are A (a, f(a)) and B (b, f(b)).

Slope of line AB, m1 =

Now there is a point c ϵ (a, b) where the slope is parallel to AB.

Since f(x) is continuous and differentiable in (a, b), we will get a tangent at point c.

Let, the slope in point c = PQ =

PQ is parallel to AB.

Therefore,

m1

∴ (proved)

**Expansion of Functions:**

1. Find the value of c in the mean value theorem.

If,

Solution:

Given that,

,

Now,

Since, , the +ve sign is to be rejected

(Ans.)

2. In the mean value theorem,

If,

Solution:

Given that,

Here, a=1, h =3

,

(Ans.)

3. In the mean value theorem,

if

Solution:

Given that,

Given equation is,

When h=7,

(Ans.)

4. In the mean value theorem,

and a = 0, h = 3. Show that has got two values and find them.

Solution:

Given,

; a = 0, h = 3

Given equation is,

Thus, θ has got two values.

(showed)

**Maxima and Minima:**

1. Find for what value of x, the following expression is maximum and minimum respectively:. Find also the maximum and minimum values of the expression.

Solution:

Let,

. . . . . . . . . . . . . . .(i)

. . . . . . . . . . . . . . .(ii) [Differentiating with respect to x]

Now, when (𝑥) is a maximum or a minimum,

Or,

Or,

Or,

Or,

Or,

or

From (ii),

Again,

. . . . . . . . . . . . . . .(iii) [Differentiating with respect to x]

Now,

when, , , which is negative.

when, , , which is positive.

Hence, the given expression is maximum for 𝑥=1 and minimum for 𝑥=6.

The maximum and minimum values of the given expression are respectively,

For, ,

For, ,

(Ans.)

2. Investigate for what values of ,

Is a maximum or minimum.

Solution:

Given that,

. . . . . . . . . . . . . (i)

. . . . . . . . . . . (ii) [Differentiating with respect to x]

When *f(x)* is a maximum or a minimum,

Or,

Or,

Or,

Or,

Or,

From (ii) again, differentiating with respect to x,

. . . . . . . . . . . . . . . . (iii)

When, , which is negative and hence is a maximum value.

When, , which is positive and hence is a minimum value.

When, , , so the test fails and we have to examine higher order derivatives.

From(iii) again differentiating with respect to x,

. . . . . . . . . . . . . . (iv)

Now,

When, , , again the test fails and we have to examine higher order derivatives.

From(iv), again differentiating with respect to x,

. . . . . . . . . . . . . . .(v)

Now,

When, , , which is positive and hence is a minimum value.

Now,

For, , is a minimum value.

For, , is a maximum value.

For, , is a minimum value.

(Ans.)

3. Examine for maximum or minimum values.

Solution:

Given that,

. . . . . . . . . . . . . . . . (i)

. . . . . . . . . . . . . . . (ii) [Differentiating with respect to x]

When *f(x)* is a maximum or a minimum,

Or,

Or,

Or,

Or,

Or,

or

From (ii), again differentiating with respect to x,

. . . . . . . . . . . . . . .(iii)

Now,

when, , , which is negative.

when, , , which is positive.

Hence, the given expression is maximum for and minimum for .

The maximum and minimum values of the given expression are respectively,

For, ,

For, ,

(Ans.)

4. Find the maxima and minima of

Solution:

Let, . . . . . . . . . . . . . . . .(i)

. . . . . . . . . . .(ii) [Differentiating with respect to x]

When *f(x)* is a maximum or a minimum,

Or,

Or,

and

From(ii), again differentiating with respect to x,

. . . . . . . . . . . . . . . .(iii)

When, , then

, which is positive.

When,

, which is negative.

Hence, the given expression is maximum for and minimum for

The maximum and minimum values of the given expression are respectively,

Now,

For, ,

For, which means, , (Ans.)

5. Examine whether possesses a maximum or a minimum and determine the same.

Solution:

Let,

. . . . . . . . . . . . . . . . . . . (i)

Differentiating equation (i) with respect to

Or,. . . . . . . . . . . . (ii)

For maxima and minima we have,

Or,

Or,

Or,

Again, differentiating equation (ii) with respect to ,

Or,

(for,

When, , which is negative.

For, the function is maximum.

Now, the maximum value is .

(Ans.)

6. Find the maximum and minimum values of where,

and

Solution:

Given that,

Eliminating between the two given relations,

. . . . . . . . . . . . . . . . . .(i)

Differentiating equation (i) with respect to ,

. . . . . . . . . . . . . . . . . (ii)

Or,

For maxima and minima ,

=0

Or, =0

Or, )=0

Or, or

Again, differentiating equation (ii) with respect to

Now,

When,,

, which is negative.

When, ,

, which is positive.

Hence, the given expression is maximum for and minimum for .

The maximum and minimum values of the given expression are respectively,

For, ,

Maximum value of

For, ,

Minimum value of (Ans.)

7. Show that the maximum value of is less than its minimum value.

Solution:

Let,

………………(i)

Differentiating equation (i) with respect to (2 times)

Or,

Or,

For maxima and minima

∴

Or, x=1 or -1

when, x = 1, = = 2 which is positive

for x = 1, y is minimum.

∴minimum value of y = =

when, 1, which is negative.

for 1, y is a maximum

∴ maximum value of

∴ The maximum value of is less than its minimum value.

(showed).

8. Show that the following function possess neither a maximum nor a minimum.

(i) (ii)

(iii) (iv)

Solution:

(i)Let,

Differentiating with respect to (2 times),

For maximum and minimum value,

f ‘(x) = 0

∴

x =

=

we can see considering f ‘(x) = 0

x doesn’t have any real value,

so, doesn’t have maximum and minimum value.

(ii) Let,

Differentiating with respect to x,

∴ f ‘(x) =

for maximum and minimum values,

f ‘(x) = 0

∴ or,

∴

we can see that, considering f ‘(x) = 0 x doesn’t have any real value.

so, neither have a maximum nor a minimum value.

(iii) Let,

Differentiating with respect to x,

for, maximum and minimum value,

f ‘(x) = 0

Or,

∴

neither have a maximum nor a minimum value.

(iv) Let,

Differentiating with respect to x,

, that will not be zero for any real value of x.

so, neither have a maximum nor a minimum value.

9. Show that is a maximum when x = 1, a minimum when x = 3; neither when x = 0.

Solution:

Let,

Differentiating with respect to ,

for maximum and minimum value,

f ‘(x)= 0

∴

Or,

Or,

∴ Or,

Or,

∴ x = 1, 3

Again, differentiating with respect to ,

when, x = 1,

So, we will get maximum value of f(x) at x = 1.

at x = 3,

We will get minimum value of f(x) at x = 3.

at x = 0,

So, test fails.

We have to examine high order derivatives, 60x**2**-120x+30

at x=0,

Therefore, f(x) is neither a maximum or a minimum value when x = 0.

(showed)

**Partial Differentiation:**

1. If , then show that, .

Solution:

Given that,

L.H.S

(showed)

2. If show that, .

Solution:

Given that,

L.H.S

(showed)

3. Show that , if

Solution:

Given that,

Partially differentiating *u* with respect to *y* (2 times),

Similarly, partially differentiating *u* with respect to *y* (2 times),

L. H. S.

(showed)

4. Show that, , if

Solution:

Given that,

Partially differentiating *u* with respect to *y* (2 times),

Similarly, partially differentiating *u* with respect to *y* (2 times),

L. H. S.

(showed)

5. If, then showed that,

(i)

(ii)

(iii)

Solution:

(i)

Given that,

Partially differentiating *u* with respect to *x, y, z* respectively,

L. H. S.

(showed)

(ii)

Where ω is the imaginary root.

Partially differentiating *u* with respect to *x,* *y* and *z* respectively,

… … … … … (1)

… … … … … (2)

… … … … … (3)

Partially differentiating equation 1, 2 and 3 with respect to *x,* *y* and *z* respectively,

L. H. S.

(showed)

(iii)

Given that,

Differentiating *u* with respect to *x,* *y* and *z* respectively,

L. H. S.

(showed)

6. If, , then show that,

Solution:

Given that,

… … … … … (1)

Partially differentiating 1 with respect to *x* (2 times)

Similarly, by partially differentiating 1 with respect to *y* and *z* respectively,

L. H. S.

[]

(showed)

7. If, , then show that,

Solution:

Given that,

Partially differentiating *v* with respect to *x* (2 times)

Similarly, differentiating *v* with respect to *y* and *z* (2 times) respectively,

L. H. S.

(showed)

8. If, , then prove that,

Solution:

Given that,

Partially differentiating *u* with respect to *z*,

… … … … (1)

Partially differentiating equation 1 with respect to *y*,

… … … … (2)

Partially differentiating equation 2 with respect to *x*,

L. H. S.

(showed)

9. If, and , prove that,

Solution:

Given that,

Again, given that,

Now,

L.H.S )

= 1

(proved)

10. If,  and , then prove that,

Solution:

Given that,

Partially differentiating *u* with respect to *x,*

Similarly, partially differentiating *u* with respect to *y, z* (2 times)

L. H. S.

(proved)